

$$I = \int_0^\infty \ln\left(\frac{e^x + 1}{e^x - 1}\right) dx$$

- (A) $| > 2$ (B) $| < 1$ (C) $| = 2$ (D) $1 < | < 2$

Solution

$$I = \int_0^\infty \ln\left(\frac{1+e^{-x}}{1-e^{-x}}\right) dx$$

Let, $e^{-x} = t$

So, $-e^{-x}dx = dt$

$$\therefore dx = -\frac{dt}{t}$$

$$\begin{aligned} I &= \int_1^0 \ln\left(\frac{1+t}{1-t}\right) \left(-\frac{1}{t}\right) dt = \int_0^1 \frac{1}{t} \ln\left(\frac{1+t}{1-t}\right) dt \\ &= \int_0^1 \frac{\ln(1+t)}{t} dt - \int_0^1 \frac{\ln(1-t)}{t} dt \\ &= \int_0^1 \frac{t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots}{t} dt - \int_0^1 \frac{-t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots}{t} dt \\ &= \int_0^1 \left(1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \dots\right) dt - \int_0^1 \left(-1 - \frac{t}{2} - \frac{t^2}{3} - \frac{t^3}{4} - \dots\right) dt \\ &= \left. \left(t - \frac{t^2}{2^2} + \frac{t^3}{3^2} - \frac{t^4}{4^2} + \dots \right) \right|_0^1 + \left. \left(t + \frac{t^2}{2^2} + \frac{t^3}{3^2} + \frac{t^4}{4^2} + \dots \right) \right|_0^1 \\ &= \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right) + \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right) \\ &= 2 \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right) \end{aligned}$$

Hence, Option (A).