

$$\text{Given, } f(x) = \left(\frac{x}{\pi}\right)^x + \left(\frac{\pi}{x}\right)^x$$

$$\int f(x)dx + \int f(x)\ln x dx - \int f(x)\ln \pi dx = ?$$

Solution

$$\text{Combining given three integrals yields } \int f(x)[1 + \ln x - \ln \pi]dx$$

$$= \int f(x)(\ln e + \ln x - \ln \pi)dx$$

$$= \int \left[\left(\frac{x}{\pi}\right)^x + \left(\frac{\pi}{x}\right)^x \right] \left(\ln \frac{ex}{\pi} \right) dx = I$$

$$\text{Let, } \left(\frac{x}{\pi}\right)^x = t$$

$$\therefore x \ln \frac{x}{\pi} = \ln t$$

$$\therefore x(\ln x - \ln \pi) = \ln t$$

$$\text{Differentiation yields } x \cdot \frac{1}{x} + \ln x - \ln \pi = \frac{1}{t} \cdot \frac{dt}{dx}$$

$$\therefore (1 + \ln x - \ln \pi)dx = \frac{dt}{t}$$

$$\therefore (\ln e + \ln x - \ln \pi)dx = \frac{dt}{t}$$

$$\Rightarrow \ln \left(\frac{ex}{\pi} \right) dx = \frac{dt}{t}$$

$$I = \int \left(t + \frac{1}{t} \right) \frac{dt}{t}$$

$$= \int \left(1 + \frac{1}{t^2} \right) dt$$

$$= t - \frac{1}{t} + C$$

$$= \left(\frac{x}{\pi} \right)^x - \left(\frac{\pi}{x} \right)^x + C$$