For $a,b,c\in R^{+}$ and $\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}=1$, show that $abc\le\frac{1}{8}$.

Solution

We have, $\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}=1$

$$\therefore \frac{1}{1+\frac{1}{a}}+\frac{1}{1+\frac{1}{b}}+\frac{1}{1+\frac{1}{c}}=1$$

Or, $\frac{1}{1+A}+\frac{1}{1+B}+\frac{1}{1+C}=1; A=\frac{1}{a}, B=\frac{1}{b}, C=\frac{1}{c}; A,B,C\in R^{+}$

$$\therefore (1+B)(1+C)+(1+C)(1+A)+(1+A)(1+B)=(1+A)(1+B)(1+C)$$

$$\Rightarrow 1+C+B+BC+1+A+C+CA+1+B+A+AB=1+C+B+BC+A+AC+AB+ABC$$

$$\Rightarrow 1+C+1+B+A=ABC$$

$$\Rightarrow A+B+C=ABC-2$$

Using $A.M.\ge G.M.$, $\frac{A+B+C}{3}\ge (ABC)^{1/3}$

$$\therefore \frac{ABC-2}{3}\ge (ABC)^{1/3}$$

$\therefore \left(\frac{ABC-2}{3}\right)^{3}\ge ABC$ Or $\left(\frac{k-2}{3}\right)^{3}\ge k$ Where ABC = k

$$\therefore k^{3}-6k^{2}+12k-8\ge 27k$$

$$\Rightarrow k^{3}-6k^{2}-15k-8\ge 0$$

$$\Rightarrow k^{2}(k-8)+2k(k-8)+(k-8)\ge 0$$

$$\Rightarrow (k-8)(k^{2}+2k+1)\ge 0$$

$\because k=ABC>0$, Thus $k^{2}+2k+1>0$

$\therefore (k-8)(+ve)\ge 0$ Or $k-8\ge 0$

$\therefore k\ge 8$ Or $ABC\ge 8$

$\therefore \frac{1}{abc}\ge 8$ Or $abc\le\frac{1}{8}$