$$y = \frac{x^4 - x^2}{x^6 + 2x^3 - 1}, x > 1$$
$$y_{\max} = ?$$

Solution

$$y = \frac{x - \frac{1}{x}}{x^3 + 2 - \frac{1}{x^3}}$$

Let, $x - \frac{1}{x} = t \quad (\because x > 1, t > 0)$

$$\Rightarrow x^3 - \frac{1}{x^3} - 3x^2\frac{1}{x} + 3x\frac{1}{x^2} = t^3$$

$$\Rightarrow x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) = t^3$$

$$\Rightarrow x^3 - \frac{1}{x^3} = t^3 + 3t$$

$$y = \frac{t}{t^3 + 3t + 2} \quad (\because t > 0, y > 0)$$

$y \to \max.$ when $\frac{t^3 + 3t + 2}{t} \to \min.$

Let, u $= \frac{t^3 + 3t + 2}{t} = t^2 + \frac{2}{t} + 3$

$u \to \min.$ when $t^2 + \frac{2}{t} = v \to \min.$

$$\frac{dv}{dt} = 2t - \frac{2}{t^2}$$

$\frac{dv}{dt} = 0,$ when $t = 1$

$\frac{d^2v}{dt^2} = 2 + \frac{4}{t^3} > 0$ for $t = 1$

Hence, minima at t=1

$$y_{\max} = \left.\frac{t}{t^3 + 3t + 2}\right|_{t=1} = \frac{1}{6}$$