If A, B and C are the angles of a triangle, and

$$\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\ge m,$$

then

(A) $m=2^2$ (B) $m=2^3$ (C) $m=2^4$ (D) $m=2^5$

Solution

We have, $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$

$$\therefore \tan\left(\frac{A}{2}+\frac{B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}$$

$$\therefore \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{1}{\tan\frac{C}{2}}$$

$$\therefore \sum\tan\frac{A}{2}\tan\frac{B}{2}=1$$

Now, $\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)^2\ge 0$

$$\therefore \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}-\tan\frac{A}{2}\tan\frac{B}{2}-\tan\frac{B}{2}\tan\frac{C}{2}-\tan\frac{C}{2}\tan\frac{A}{2}\ge 0$$

$$\therefore \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge\sum\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\therefore \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge 1$$

$$\therefore \left(\sec^2\frac{A}{2}-1\right)+\left(\sec^2\frac{B}{2}-1\right)+\left(\sec^2\frac{C}{2}-1\right)\ge 1$$

$$\therefore \sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\ge 4$$

Hence, (A)