Let $I(x)=\int\left(\frac{\sin^2 x}{\cos^{14} x}\right)^{1/3} dx, I(0)=0$ then $I\left(\frac{\pi}{4}\right)$ is equal to

(A) $\frac{3}{5}$ (B) $\frac{3}{11}$ (C) $\frac{18}{55}$ (D) $\frac{48}{55}$

Solution

$$I(x)=\int\left(\tan^2 x\sec^{12} x\right)^{1/3} dx=\int\tan^{2/3} x\sec^4 x dx$$

Let, $\tan x=t$

Then, $\sec^2 x dx=dt$

$$\therefore I(t)=\int t^{2/3}(1+t^2)dt=\int(t^{2/3}+t^{8/3})dt=\frac{t^{5/3}}{5/3}+\frac{t^{11/3}}{11/3}+C$$

$$\therefore I(x)=\frac{3}{5}\tan^{5/3} x+\frac{3}{11}\tan^{11/3} x+C$$

$I(0)=0$ yields $C=0$

$$\therefore I(x)=\frac{3}{5}\tan^{5/3} x+\frac{3}{11}\tan^{11/3} x$$

$$\therefore I\left(\frac{\pi}{4}\right)=\frac{3}{5}+\frac{3}{11}=\frac{48}{55}$$

Hence, (D)