Let $y = f(x) = \frac{x+2}{2x+3}$, y > 0. If

$$I = \int \sqrt{\frac{f(x)}{x^2}}dx = \frac{1}{\sqrt{2}} g\left(\frac{1+\sqrt{2f(x)}}{1-\sqrt{2f(x)}}\right) - \frac{1}{\sqrt{3}} h\left(\frac{\sqrt{3f(x)}+\sqrt{2}}{\sqrt{3f(x)}-\sqrt{2}}\right) + C$$. Then,

(A) $g(x) = \tan^{-1} x, h(x) = \ln|x|$

(B) $g(x) = \ln|x|, h(x) = \tan^{-1} x$

(C) $g(x) = \tan^{-1} x, h(x) = \tan^{-1} x$

(D) $g(x) = \ln|x|, h(x) = \sqrt{2}\ln|x|$

Solution

Let $\sqrt{f(x)} = t$ or $t^2 = \frac{x+2}{2x+3}$

$$\therefore x = \frac{2-3t^2}{2t^2-1}$$

$$\therefore dx = -\frac{2t}{(2t^2-1)^2}dt$$

Given integral, $I = \int \frac{\sqrt{y}}{x}dx = \int \frac{t}{\frac{2-3t^2}{2t^2-1}} \frac{-2tdt}{(2t^2-1)^2} = 2\int \frac{t^2dt}{(3t^2-2)(2t^2-1)}$

$$\therefore I = 2\int \frac{2}{3t^2-2} - \frac{1}{2t^2-1}dt$$

$$\therefore I = -\sqrt{\frac{2}{3}}\ln\left|\frac{\sqrt{3}t+\sqrt{2}}{\sqrt{3}t-\sqrt{2}}\right| + \frac{1}{\sqrt{2}}\ln\left|\frac{1+\sqrt{2}t}{1-\sqrt{2}t}\right| + C$$

$$\therefore I = \frac{1}{\sqrt{2}}\ln\left|\frac{1+\sqrt{2f(x)}}{1-\sqrt{2f(x)}}\right| - \sqrt{\frac{2}{3}}\ln\left|\frac{\sqrt{3f(x)}+\sqrt{2}}{\sqrt{3f(x)}-\sqrt{2}}\right| + C$$

So, $g(x) = \ln|x|$ & $h(x) = \sqrt{2}\ln|x|$

Hence, (D)