$$I=\int_0^1 \frac{x^9}{\sqrt{1-x^4}}dx=$$

(A) $\frac{3\pi}{4}$ (B) $\frac{\pi}{16}$ (C) $\frac{3\pi}{32}$ (D) $\frac{\pi}{8}$

Solution

Let $x^2=t$

$\therefore 2xdx=dt$

$$I=\int_0^1 \frac{t^4}{2\sqrt{1-t^2}}dt$$

$$I=\frac{1}{2}\int_0^1 \frac{(t^4-1)+1}{\sqrt{1-t^2}}dt=\frac{1}{2}\int_0^1 \frac{(t^2-1)(t^2+1)}{\sqrt{1-t^2}}dt+\frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-t^2}}dt$$

$$\therefore I=-\frac{1}{2}\int_0^1 \sqrt{1-t^2}(t^2+1)dt+\frac{1}{2}\sin^{-1}t\Big|_0^1$$

$$\therefore I=-\frac{1}{2}\int_0^1 t^2\sqrt{1-t^2}dt-\frac{1}{2}\int_0^1 \sqrt{1-t^2}dt+\frac{\pi}{4}$$

$$\therefore I=\frac{\pi}{4}-\frac{1}{2}\left(\frac{t\sqrt{1-t^2}}{2}+\frac{1}{2}\sin^{-1}t\right)\Bigg|_0^1-\frac{1}{2}\int_0^1 t^2\sqrt{1-t^2}dt$$

$$\therefore I=\frac{\pi}{4}-\frac{1}{2}\times\frac{\pi}{4}-\frac{1}{2}\int_0^1 t^2\sqrt{1-t^2}dt$$

Let $t=\sin\theta$

$\therefore dt=\cos\theta d\theta$

$$\therefore I=\frac{\pi}{4}-\frac{\pi}{8}-\frac{1}{2}\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta$$

$$\therefore I=\frac{\pi}{8}-\frac{1}{2}\times\frac{1}{4\times 2}\times\frac{\pi}{2}=\frac{\pi}{8}-\frac{\pi}{32}=\frac{3\pi}{32}$$

Hence, (C)