$f(x+y)=f(x)+f(y)+xy$ for all values of $x, y \in \mathbb{R}$. If $f(4)=10$, f(100) is:

(A) $1+2+3+\ldots\ldots\ldots\ldots\ldots+100$
(B) < 0
(C) 0
(D) None of the options given

Solution

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x)+f(h)+xh-f(x)}{h}$$

So, $f'(x)=\lim_{h\to 0}\frac{f(h)+xh}{h}$

Putting $y=0$ in $f(x+y)=f(x)+f(y)+xy$ it can be seen that $f(0)=0$.

Thus, the limit $\lim_{h\to 0}\frac{f(h)+xh}{h}$ has $\left[\frac{0}{0}\right]$ form.

$\therefore f'(x)=\lim_{h\to 0}\frac{f'(h)+x}{1}$ [L. H. Rule]

$\therefore f'(x)=x+\lim_{h\to 0}f'(h)=x+k$

$\therefore f(x)=\int(x+k)dx=\frac{x^2}{2}+kx+C$

$\because f(0)=0, C=0$

$\therefore f(x)=\frac{x^2}{2}+kx$

Given, $f(4)=10=\frac{16}{2}+4k$

$\therefore k=\frac{1}{2}$

So, $f(x)=\frac{x^2}{2}+\frac{1}{2}x=\frac{x(x+1)}{2}$

$$f(100)=\frac{100\times(100+1)}{2}=1+2+3+\ldots\ldots\ldots+100$$

Hence, (A)