Evaluate the integral $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ and hence prove that $\pi < \frac{22}{7}$.

*Solution*

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 \frac{x^4({}^4C_0 - {}^4C_1 x + {}^4C_2 x^2 - {}^4C_3 x^3 + {}^4C_4 x^4)}{1+x^2}dx$$

$$= \int_0^1 \frac{x^4(1-4x+6x^2-4x^3+x^4)}{1+x^2}dx$$

$$= \int_0^1 x^4 \frac{[-4x(1+x^2)+1+6x^2+x^4]}{1+x^2}dx$$

$$= \int_0^1 x^4(-4x)dx + \int_0^1 x^4 \frac{(1+6x^2+x^4)}{1+x^2}dx$$

$$= -4\times\frac{1}{6} + \int_0^1 x^4 \frac{1+5x^2+x^2(1+x^2)}{1+x^2}dx$$

$$= -\frac{2}{3} + \int_0^1 x^6 dx + \int_0^1 x^4 \frac{1+5x^2}{1+x^2}dx$$

$$= -\frac{2}{3} + \frac{1}{7} + \int_0^1 x^4 \frac{5(1+x^2)-4}{1+x^2}dx$$

$$= -\frac{2}{3} + \frac{1}{7} + \int_0^1 5x^4 dx - 4\int_0^1 \frac{x^4}{1+x^2}dx$$

$$= -\frac{2}{3} + \frac{1}{7} + 1 - 4\int_0^1 \frac{x^4-1+1}{1+x^2}dx$$

$$= \frac{1}{3} + \frac{1}{7} - 4\int_0^1 \frac{(x^2+1)(x^2-1)+1}{1+x^2}dx$$

$$= \frac{1}{3} + \frac{1}{7} - 4\int_0^1 (x^2-1)dx - 4\int_0^1 \frac{1}{1+x^2}dx$$

$$= \frac{1}{3} + \frac{1}{7} - 4\left(\frac{1}{3}-1\right)dx - 4\frac{\pi}{4} = \frac{1}{3} + \frac{1}{7} + \frac{8}{3} - \pi = 3 + \frac{1}{7} - \pi = \frac{22}{7} - \pi$$

Now, $f(x) = \frac{x^4(1-x)^4}{1+x^2}$ has even powers and having +ve area in the interval 0 to 1.

So, $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \frac{22}{7} - \pi > 0$ or $\frac{22}{7} > \pi$ or $\pi < \frac{22}{7}$

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