

### Problem

Evaluate  $\lim_{x \rightarrow \infty} \frac{x^3}{\sqrt{x^2 - a^2}} - \frac{x^3}{\sqrt{x^2 + a^2}}$

### Solution

$$\begin{aligned}
 & \text{The given limit can be written as } \lim_{x \rightarrow \infty} x^3 \cdot \frac{\sqrt{x^2 + a^2} - \sqrt{x^2 - a^2}}{\sqrt{x^2 - a^2} \cdot \sqrt{x^2 + a^2}} \\
 &= \lim_{x \rightarrow \infty} x^3 \cdot \frac{\sqrt{x^2 + a^2} - \sqrt{x^2 - a^2}}{\sqrt{x^2 - a^2} \cdot \sqrt{x^2 + a^2}} \times \frac{\sqrt{x^2 + a^2} + \sqrt{x^2 - a^2}}{\sqrt{x^2 + a^2} + \sqrt{x^2 - a^2}} \\
 &= \lim_{x \rightarrow \infty} x^3 \cdot \frac{2a^2}{\sqrt{x^2 - a^2} \cdot \sqrt{x^2 + a^2} \cdot (\sqrt{x^2 + a^2} + \sqrt{x^2 - a^2})} \\
 &= 2a^2 \lim_{x \rightarrow \infty} x^3 \cdot \frac{1}{x \sqrt{1 - \frac{a^2}{x^2}} \cdot x \sqrt{1 + \frac{a^2}{x^2}} \cdot x \left( \sqrt{1 + \frac{a^2}{x^2}} + \sqrt{1 - \frac{a^2}{x^2}} \right)} \\
 &= 2a^2 \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 - \frac{a^2}{x^2}} \cdot \sqrt{1 + \frac{a^2}{x^2}} \left( \sqrt{1 + \frac{a^2}{x^2}} + \sqrt{1 - \frac{a^2}{x^2}} \right)} \\
 &= 2a^2 \times \frac{1}{2} = a^2
 \end{aligned}$$