

Let $f: \left[\frac{1}{2}, 1\right] \rightarrow \mathbb{R}$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of

$\int_{1/2}^1 f(x) dx$ lies in the interval

(A) $(2e - 1, 2e)$

(B) $(e - 1, 2e - 1)$

(C) $\left(\frac{e-1}{2}, e-1\right)$

(D) $\left(0, \frac{e-1}{2}\right)$

Solution

We have, $f'(x) < 2f(x)$

$$\therefore \frac{f'(x)}{f(x)} < 2 \quad [\because f(x) > 0]$$

$$\int_{1/2}^x \frac{f'(x)}{f(x)} dx < \int_{1/2}^x 2 dx \quad [x > 0]$$

$$\ln f(x) \Big|_{1/2}^x < 2x \Big|_{1/2}^x$$

$$\Rightarrow \ln f(x) - \ln f\left(\frac{1}{2}\right) < 2\left(x - \frac{1}{2}\right)$$

$$\Rightarrow \ln f(x) < 2x - 1 \quad \left[\because f\left(\frac{1}{2}\right) = 1\right]$$

$$\Rightarrow f(x) < e^{2x-1}$$

Also, it is given that $f(x) > 0$.

So, $0 < f(x) < e^{2x-1}$

$$\int_{1/2}^1 0 dx < \int_{1/2}^1 f(x) dx < \int_{1/2}^1 e^{2x-1} dx$$

$$0 < \int_{1/2}^1 f(x) dx < \frac{e-1}{2}$$

Hence, D.